

# Relativity and $c/\sqrt{3}$

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## Abstract

We define the critical coordinate velocity  $v_c$ . A particle moving radially in Schwarzschild background with this velocity,  $v_c = c/\sqrt{3}$ , is neither accelerated, nor decelerated if gravitational field is weak,  $r_g \ll r$ , where  $r_g$  is the gravitational radius, while  $r$  is the current one. We find that the numerical coincidence of  $v_c$  with velocity of sound in ultrarelativistic plasma,  $u_s$ , is accidental, since two velocities are different if the number of spatial dimensions is not equal to 3.

## To Holger Nielsen

We dedicate this note to our friend Holger Nielsen on the occasion of his 60th birthday. It has been a great pleasure to discuss with him exiting physical ideas at ITEP, CERN, the Niels Bohr Institute and in other places around the world. We wish to Holger many new discoveries and a long happy life.

## 1 Motivation

According to General Relativity (GR) clocks run slowly in the presence of gravitational field, as a result, the coordinate velocity of photons decreases. This is the reason for the delay of radar echo from inner planets predicted and measured by I. Shapiro [1]. Propagation of ultrarelativistic particles is described similarly to that of photons. That is why the retardation must take place not only for photons but also for ultrarelativistic particles. In this respect the latter drastically differ [2] from nonrelativistic bodies, velocity of which evidently increases when they are falling radially onto a gravitating body (e.g., onto the Sun). Obviously, there should be some intermediate velocity  $v_c$  which remains constant for a particle falling in gravitational field of the Sun (or another star). The numerical value  $v_c = c/\sqrt{3}$  will be found in Sect. 2. When a particle moves radially with this velocity in weak field it “ignores” gravity: it is neither accelerated, nor decelerated. For nonradial trajectories gravity is never ignored: the trajectories are bent for any velocity.

It is well known that  $u_s = c/\sqrt{3}$  is the speed of sound in ultrarelativistic plasma and the question arises whether the equality  $u_s = v_c$  has some physical reason, or it is a numerical coincidence. To answer this question we find in Sect. 3 expressions for  $v_c$  and  $u_s$  in spaces with number of dimensions  $n$  different from 3. Since for  $n \neq 3$  we get  $v_c \neq u_s$  we come to the conclusion that their coincidence at  $n = 3$  does not have deep physical reason.

## 2 Derivation of $v_c = c/\sqrt{3}$

To simplify formulas, we put light velocity  $c = 1$ , restoring it when it is necessary. In what follows  $G$  is gravitational constant; gravitational radius  $r_g$  of an object with mass  $M$  equals

$$r_g = 2GM . \quad (1)$$

Let us start from definitions used in GR. The expression for interval in the case of radial motion ( $d\theta = d\varphi = 0$ ) has the well known Schwarzschild form:

$$ds^2 = g_{00}dt^2 - g_{rr}dr^2 \equiv d\tau^2 - dl^2 , \quad (2)$$

where  $g_{00} = (g_{rr})^{-1} = 1 - \frac{r_g}{r}$ . The local velocity  $v$  of a particle measured by a local observer at rest is:

$$v = \frac{dl}{d\tau} = \left( \frac{g_{rr}}{g_{00}} \right)^{1/2} \frac{dr}{dt} = \frac{1}{g_{00}} \frac{dr}{dt} , \quad (3)$$

while observer at infinity, where  $g_{00}(\infty) = g_{rr}(\infty) = 1$ , measures the so-called coordinate velocity at  $r$ :

$$v = \frac{dr}{dt} = v \left( \frac{g_{00}}{g_{rr}} \right)^{1/2} = g_{00}v . \quad (4)$$

In order to determine the time of radial motion from  $a$  to  $b$ , the infinitely distant observer should calculate the integral

$$t = \int_a^b \frac{dr}{v} , \quad (5)$$

that is why the coordinate velocity is relevant for radar echo.

For a particle moving in static gravitational field one can introduce conserved energy (see ref. [3], eq. 88.9):

$$E = \frac{m\sqrt{g_{00}}}{\sqrt{1-v^2}} . \quad (6)$$

The expression for  $E$  through  $v$ :

$$E = \frac{m\sqrt{g_{00}}}{\sqrt{1-(v/g_{00})^2}} \quad (7)$$

allows us to determine  $v(r)$  from the energy conservation:

$$E(r = \infty) = E(r) , \quad (8)$$

$$v^2 = g_{00}^2 - g_{00}^3 + g_{00}^3 v_\infty^2 = g_{00}^2 [1 - g_{00}(1 - v_\infty^2)] . \quad (9)$$

For the local velocity  $v$  measured by a local observer we obtain:

$$v^2 = 1 - g_{00}(1 - v_\infty^2) , \quad (10)$$

so, while  $v$  always increases for a falling massive particle, reaching  $c$  at  $r = r_g$ , the behaviour of  $v$  is more complicated. Substituting  $g_{00} = 1 - \frac{r_g}{r}$  into (9), we get for weak gravitational field ( $r \gg r_g$ ):

$$v^2 = v_\infty^2 + \frac{r_g}{r}(1 - 3v_\infty^2) . \quad (11)$$

For the motion of a nonrelativistic particle ( $v_\infty \ll 1$ ) the well-known expression is reproduced:

$$v^2 = v_\infty^2 + \frac{2MG}{r} . \quad (12)$$

For  $v_\infty = v_c = 1/\sqrt{3}$  the coordinate velocity of particle does not change, while it grows for  $v_\infty < v_c$  and diminishes for  $v_\infty > v_c$ . At  $r = 3r_g$  according to eq.(11) the coordinate velocity becomes equal to  $v_c$ . However, for  $r = 3r_g$  our weak field approximation fails.

Let us dispose of the assumption of the weak field. Coming back to expression (9) and substituting there  $g_{00} = 1 - \frac{r_g}{r}$ , we observe that for  $v_\infty > v_c$  the coordinate velocity always diminishes and becomes zero at  $r = r_g$ , while in the case  $v_\infty < v_c$  it grows up to the value

$$v_{\max}^2 = 4/(27(1 - v_\infty^2)^2) , \quad (13)$$

which is reached at

$$r_0 = \frac{3(1 - v_\infty^2)}{(1 - 3v_\infty^2)} r_g , \quad (14)$$

and after that diminishes to zero at  $r = r_g$ . It is interesting to note that the velocity  $v$  measured by local observer equals  $v_c$  at the point where  $v = v_{\max}$ .

Thus, if the coordinate velocity is only mildly relativistic,  $v_\infty > c/\sqrt{3}$ , then  $v$  already decreases at the free fall.

As an example of a non-radial motion let us consider the deflection of light from a star by the Sun and compare it with the deflection of a massive particle. It is well known that the angle of deflection  $\theta$  of photons grazing the Sun is given by

$$\theta_\gamma = \frac{2r_g}{R_\odot} \quad (15)$$

where  $R_\odot$  is the radius of the Sun. In the case of massive particles the deflection angle is larger:

$$\theta = \theta_\gamma(1 + \beta^{-2}) , \quad (16)$$

where  $\beta \equiv v_\infty/c < 1$ . (See ref.[4], eq. 25.49, and ref.[5], problem 15.9, eq.13.)

### 3 Speed of sound $u_s$ and critical speed $v_c$ in $n$ dimensions

For ultrarelativistic plasma with equation of state  $P = e/3$ , where  $P$  is pressure and  $e$  is energy density (including mass density), we have for the speed of sound  $u_s$ :

$$u_s^2 = c^2 \left. \frac{\partial P}{\partial e} \right|_{\text{ad}} = \frac{c^2}{3} , \quad (17)$$

We use eq. 134.14 of ref. [6], and correct misprint there, or eq. 126.9 from [7]; “ad” means adiabatic, i.e. for constant specific entropy. In order to obtain the expression for  $u_s$  in the case when  $n \neq 3$ , where  $n$  is the number of spatial dimensions, let us start with equation of state.

One can use a virial theorem to connect pressure  $P$  and thermal energy  $\mathcal{E}$  of an ideal gas using classical equations of particle motion (cf. [8]). We have for a particle with momentum  $\mathbf{p}$  and a Hamiltonian  $H$ :

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} , \quad (18)$$

hence,

$$\mathbf{q}\dot{\mathbf{p}} = -\mathbf{q}\frac{\partial H}{\partial \mathbf{q}} = \mathbf{q}\mathbf{F} , \quad (19)$$

where  $\mathbf{F}$  is the force acting on the particle. Let us average over time  $t$ :

$$\langle \dots \rangle \equiv \frac{1}{t} \int_0^t \dots d\bar{t} . \quad (20)$$

Integrating by parts we get:

$$\langle \mathbf{q}\dot{\mathbf{p}} \rangle = -\langle \dot{\mathbf{q}}\mathbf{p} \rangle = \langle \mathbf{q}\mathbf{F} \rangle . \quad (21)$$

For non-relativistic (NR) particles

$$\dot{\mathbf{q}}\mathbf{p} = 2E_{\text{kin}} = \mathbf{p}^2/m . \quad (22)$$

For extremely relativistic (ER) particles

$$\dot{\mathbf{q}}\mathbf{p} = E_{\text{kin}} = c|\mathbf{p}| . \quad (23)$$

Now for  $N$  particles in a gas

$$-\sum_{i=1}^N \langle \dot{\mathbf{q}}_i \mathbf{p}_i \rangle = \sum_{i=1}^N \langle \mathbf{q}_i \mathbf{F}_i \rangle . \quad (24)$$

(By the way,  $-\frac{1}{2} \sum_i \langle \mathbf{q}_i \mathbf{F}_i \rangle$  is called the *virial*.) If the gas is ideal (i.e. non-interacting particles), then the force  $\mathbf{F}$  is non-zero only at the collision of a particle with the wall, and the virial reduces to an integral involving pressure:

$$-\sum_{i=1}^N \langle \dot{\mathbf{q}}\mathbf{p} \rangle = -\int P \mathbf{n} \mathbf{q} dS = -P \int \text{div } \mathbf{q} dV = -3PV , \quad (25)$$

where  $\mathbf{n}$  is a unit vector normal to the wall area element  $dS$  and the Gauss theorem is used for transforming the surface integral to the volume one. So, since the thermal energy  $\mathcal{E}$  (not including mass) is just the total kinetic energy of molecules,

$$\text{NR : } 2\mathcal{E} = 3PV, \quad P = 2\mathcal{E}/(3V) , \quad (26)$$

$$\text{ER : } \mathcal{E} = 3PV, \quad P = \mathcal{E}/(3V) \equiv e/3 . \quad (27)$$

The last equality holds since in extremely relativistic case  $E_{\text{kin}} \gg m$ . We see that 3 here is due to  $\text{div } \mathbf{q} = 3$ , i.e. the dimension of our space.

In a space of  $n$  dimensions, following the same lines, we get  $\text{div } \mathbf{q} = n$ , so  $P = e/n$  and for ER gas we obtain:

$$u_s = c/\sqrt{n} . \quad (28)$$

Here we should use  $n$ -volume  $V_n$  instead of  $V \equiv V_3$  and postulate the first law of thermodynamics for adiabatic processes to be  $d\mathcal{E} + PdV_n = 0$ , so pressure would be the force per unit  $V_{n-1}$  – the boundary of  $V_n$ .

The same equation of state follows from consideration of the stress tensor  $T_{ik}$  of ultrarelativistic plasma, which is diagonal and traceless in a rest frame of plasma:  $T_{00} = e$ ,  $T_{ii} \equiv P = e/n$ .

In order to find  $v_c$  in the case  $n \neq 3$  we need an  $n+1$ -dimensional spherically-symmetric static generalization of the  $3+1$ -dimensional Schwarzschild metric which was found by Tangherlini [9]. (See refs.[10] for the details of aspherical and time-dependent black holes metrics in higher dimensional spacetimes). The line element of the  $n+1$ -dimensional Schwarzschild metric is

$$ds^2 = \left(1 - \frac{r_{gn}^{n-2}}{r^{n-2}}\right) dt^2 - \left(1 - \frac{r_{gn}^{n-2}}{r^{n-2}}\right)^{-1} dr^2 - r^2 d\Omega_{n-1}^2, \quad (29)$$

where  $d\Omega_{(n-1)}$  is the line element on the unit  $(n-1)$ -sphere and the gravitational radius  $r_{gn}$  is related to the black hole mass  $M$ :

$$r_{gn}^{n-2} = \frac{16\pi G_n M}{(n-1) A_{n-1}}. \quad (30)$$

Here  $A_{n-1}$  denotes the area of a unit  $n-1$  sphere, which is  $2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$  (for  $n=3$ :  $\Gamma(3/2) = \sqrt{\pi}/2$ ,  $A_2 = 4\pi$ ). We consider the spaces with  $n \geq 3$ . The factor in the definition of  $r_{gn}$  is taken from refs.[10]. The form of the metric (29) is very easy to guess. In weak field limit, when  $g_{00} \rightarrow 1 + 2\varphi$  we have

$$\varphi = -\frac{r_{gn}^{n-2}}{2r^{n-2}} \quad \text{for } r \rightarrow \infty. \quad (31)$$

This leads in a natural way to the gravitational acceleration  $\mathbf{g}$  with the radial component

$$g_n = -\frac{\partial\varphi}{\partial r} = -\frac{(n-2)r_{gn}^{n-2}}{2r^{n-1}} \quad \text{for } r \rightarrow \infty, \quad (32)$$

which implies the constant flux of the acceleration  $\mathbf{g}$  equal to

$$A_{n-1}r^{n-1}g_n = 8\pi \frac{n-2}{n-1} G_n M \quad (33)$$

through a sphere of area  $A_{n-1}r^{n-1}$  at large  $r$ . It is not hard to verify that the Ricci tensor  $R_{ik}$  is zero for the metric (29), that is the metric (29) satisfies Einstein equations in vacuum and describes a spherically symmetric spacetime outside a spherical gravitating body.

One should remember that the dimension of  $G_n$  depends on  $n$ . It is clear in the weak field limit from eqs. (31) and (30), since the dimension of  $[\varphi]$  is the square of velocity, i.e. zero for  $c=1$ , and hence

$$[G_n] = \frac{L^{n-2}}{M}, \quad (34)$$

or  $[G_n] = L^{n-1}$ , if  $[M] = L^{-1}$ .

For the coordinate velocity of a radially falling particle we get from eq. (9) for weak gravitational field:

$$v^2 = v_\infty^2 + \left(\frac{r_{gn}}{r}\right)^{n-2} (1 - 3v_\infty^2) \quad (35)$$

instead of eq. (11). We see that in the case of  $n$ -dimensional space the expression for  $v_c$  remains the same,  $v_c = c/\sqrt{3}$ . Number “3” here is not due to the dimension of space, it is simply due to cubic polynomial in (9).

## 4 Conclusions and Acknowledgements

The speed of sound in relativistic, radiation dominated plasma depends on the dimension of space, while the critical velocity  $v_c = c/\sqrt{3}$  in the Schwarzschild metric is the same for any dimension.

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PS. After the first version of this paper appeared on the web we recieved an e-mail from Stanley Deser and Bayram Tekin in which it was pointed out that critical velocity in weak field has been considered by M.Carmeli: *Lettere al Nuovo Cimento* 3 (1972)379 and in his book “Classical Fields: General Relativity and Gauge Theory”, John Wiley and Sons, Inc 1982 New York . We thank them for this bitter remark.

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